

SOLVING ONE-VARIABLE EQUATIONS IN FREE GROUPS

DIMITRI BORMOTOV ROBERT GILMAN ALEXEI MYASNIKOV

ABSTRACT. Equations in free groups have become prominent recently in connection with the solution to the well known Tarski Conjecture. Results of Makanin and Rasborov show that solvability of systems of equations is decidable and there is a method for writing down in principle all solutions. However, no practical method is known; the best estimate for the complexity of the decision procedure is P-space.

The special case of one variable equations in free groups has been open for a number of years, although it is known that the solution sets admit simple descriptions. We use cancellation arguments to give a short and direct proof of this result and also to give a practical polynomial time algorithm for finding solution sets. One variable equations are the only general subclass of equations in free groups for which such results are known.

We improve on previous attempts to use cancellation arguments by employing a new method of reduction motivated by techniques from formal language theory. Our paper is self contained; we assume only knowledge of basic facts about free groups.

1. INTRODUCTION

A one variable equation $E(x) = 1$ of degree d in a finitely generated free group F is an expression of the form

$$(1) \quad u_0 x^{\varepsilon_0} u_1 x^{\varepsilon_1} \dots u_{d-1} x^{\varepsilon_{d-1}} = 1$$

composed of elements $u_i \in F$, integers $\varepsilon_i = \pm 1$ and a symbol x not in F . A solution to (1) is an element $g \in F$ such that substitution of g for x yields 1 in F .

Lyndon [17] was the first to study equations of this sort. He characterized solution sets in terms of parametric words. The parametric words involved were simplified by Lorents [19, 20] and Appel [1]. However, Lorents announced his results without proof, and Appel's published proof has a gap (see [6]). A complete proof has been provided recently by Chiswell and Remeslennikov [6].

Chiswell and Remeslennikov's novel analysis involves algebraic geometry ([2], [22].) First they describe the isomorphism types of the coordinate groups of irreducible one-variable equations over F , and then they deduce the structure of the solution sets. The latter part is easy, but the former requires sophisticated techniques involving ultrapowers and Lyndon length functions. The key point is that coordinate groups of irreducible equations

over F are subgroups of the ultrapower F^I/D of F over a countable set I with a non-principal ultrafilter D .

One can view the group F as a subgroup of F^I/D under the canonical diagonal embedding. From this point of view the coordinate groups are precisely the finitely generated subgroups of F^I/D containing F i.e., the so-called F -subgroups. In particular up to isomorphism the coordinate groups of irreducible one-variable equations over F are the subgroups of F^I/D of the form $\langle F, g \rangle$, $g \in F^I/D$.

Investigation of such F -subgroups of F^I/D is not easy and involves a careful analysis of Lyndon functions. (It might be interesting to see whether it is easier to use free actions on Λ -trees.) The computations can be simplified by employing a result from [11] which states that the coordinate groups of irreducible varieties are precisely the finitely generated F -subgroups of the free exponential Lyndon group $F^{\mathbb{Z}[t]}$. As this group is the union of an infinite ascending chain of extensions of centralizers of F [23], one can use Bass-Serre theory to study F -subgroups of $F^{\mathbb{Z}[t]}$.

Chisewell and Remeslennikov's method is very powerful and potentially useful for more than just free groups. However, it does have the disadvantage of not giving an algorithm for explicitly describing the set of solutions.

This paper is a refinement and extension of [9] where results from formal language theory are used to describe solution sets of one-variable equations in free groups. As it turns out, formal language techniques are not required; straightforward cancellation arguments suffice. It seems likely that these arguments can be extended to other groups admitting suitable (not necessarily Lyndon) length functions. The main advantage of this method is that it is short and yields a polynomial time algorithm for producing a description of all solutions. This algorithm has been implemented by the first author [4].

Theorem 1. *The solution set for a one variable equation of positive degree in a free group F is a finite union of sets $uv^i w$ where $u, v, w \in F$ and i ranges over all integers. There is a polynomial time algorithm for finding these sets.*

Let Σ be a set of free generators for F together with their inverses, and let Σ^* be the free monoid over Σ . We consider Equation (1) in terms of words in Σ^* . Each coefficient u_i is represented by a freely reduced word (also denoted u_i) in Σ^* . From this point of view $E(x) = u_0 x^{\varepsilon_1} u_1 x^{\varepsilon_2} \dots u_{d-1} x^{\varepsilon_d} u_d$ is a word in the free monoid over $\Sigma \cup \{x, x^{-1}\}$, and a solution to $E(x) = 1$ is a word $s \in \Sigma^*$ such that $E(s)$ is freely equal to the empty word. The first assertion of Theorem 1 is equivalent to saying that for some finite union of sets of words $uv^i w$ the solutions set consists of all words freely equal to elements of the finite union. A set $uv^i w$ is called a parametric word.

We assume without loss of generality that $E(x)$ is freely reduced, and call d the degree of $E(x)$. If $d = 0$, then $E(x) = u_0$. In this case the solution set is empty if $E(x) \neq 1$ and all of Σ^* if $E(x) = 1$. If the equation has degree

one, it is easy to find its unique solution. From now on we consider only equations of degree at least two.

We begin with some lemmas on cancellation, after which we find a finite number of parametric words $uv^i w$ and $uv^i wr^j s$ which contain all solutions to $E(x) = 1$ up to free equivalence. Next we show that two parameters are not required and that $uv^i w$ is either a solution for all integers i or for an effectively determined finite subset. At the end we present the algorithm and estimate its time complexity.

To explain our argument in more detail we require a few definitions. For any (word) $g \in F$ we say that the i th occurrence of g *cancels out* in $E(g)$ if there exists a way to freely reduce $E(g)$ such that all letters from g^{ϵ_i} cancel out during this reduction process.

We say that g is a *pseudo-solution* of $E(x) = 1$ if some occurrence of g cancels out in $E(g)$. Obviously every solution of $E(x) = 1$ is also a pseudo-solution of $E(x) = 1$. However, unlike solutions, pseudo-solutions admit a nice reduction theory.

Our key idea is to study pseudo-solutions of equations instead of solutions. The first result in this direction (stated in [9] in a slightly different form) reduces the situation to cubic equations. Namely, Lemma 12 shows that if g is a pseudo-solution of $E(x) = 1$ in F then g is a pseudo-solution of a cubic equation of the type

$$x^{\epsilon_{j-1}} u_j x^{\epsilon_j} u_{j+1} x^{\epsilon_{j+1}},$$

where $0 < j < d$ and indices are read modulo d (so $u_d = u_0$.) Next in Lemma 14 we show that pseudo-solutions of cubic equations are in fact pseudo-solutions of some particular quadratic equations which one can find effectively. Finally, Lemmas 6 and 7 give a precise description of pseudo-solutions of quadratic one-variable equations over F in terms of parametric words. Combining all these results we obtain description of all pseudo-solutions of $E(x) = 1$ in terms of parametric words in two parameters.

The rest of our proof explains precisely how to use only one parameter to describe solutions of $E(x) = 1$. The method of big powers (see [3]) is the key tool in the second part. This means that the argument is rather general - it works in many other groups that satisfy the big powers condition (see [16]), for example torsion-free hyperbolic groups.

One-variable equations are the only general class of equations in free groups for which a good description of solution sets as well as a practical (polynomial time) algorithm are known. In his seminal paper [21] Makanin proved decidability of the Diophantine problem in free groups F (whether or not a given equation has a solution in F); however, his original algorithm is very inefficient - not even primitive recursive (see [15]). In the fundamental paper [25] Razborov gave a description of solution sets of arbitrary equations in F . Though this description is extremely complicated, it was useful in the solution of several deep problems in group theory [12, 13, 5] including the Tarski's problems [14]. In another paper [26] Razborov showed

that, in general, there is no easy description of solutions sets of equations in F . Later, Plandowski gave a much improved P -space version of the decision algorithm for equations in free monoids [24], and Gutierrez devised a P -space algorithm for the decision problem for equations in free groups [10]. Recent results [7] due to Diekert, Gutierrez, and Hagenah, indicate that the decision problem for equations in free groups might be P -space-complete, though nothing definite has been proven so far. These results on the complexity of the decision problem for equations in free groups and for their solution sets make the existence of subclasses of equations admitting polynomial decision algorithms and descriptions of solutions sets in closed form, all the more remarkable.

2. CANCELLATION LEMMAS

As above Σ is a set of free generators and their inverses for a free group F , and Σ^* is the free monoid over Σ . Let p, q, r, s, t, u, v, w be words in Σ^* . We write $u \sim v$ if u is freely equal to v , and $u \rightarrow v$ if u can be reduced to v by cancellation of subwords aa^{-1} , $a \in \Sigma$. In particular $u \rightarrow u$. The empty word is denoted 1, and the length of u is $|u|$. Recall that for any word u there is a unique irreducible word v such that $u \rightarrow v$, and further $u \sim w$ if and only if $w \rightarrow v$.

We introduce some additional notation.

Definition 1. *Let w be any word.*

- (1) w' stands for an arbitrary prefix of w and w'' for an arbitrary suffix.
- (2) $|w|_c$ is the length of a cyclicly reduced word conjugate to w .

Lemma 1. *If $v \rightarrow u$ and $u = u_1u_2 \cdots u_m$, then $v = v_1v_2 \cdots v_m$ with $v_i \rightarrow u_i$.*

Proof. Use induction on n , the number of cancellations necessary to reduce v to u . If $n = 0$, then $u = v$ and there is nothing to prove. Otherwise let the first reduction be $v \rightarrow w$. By induction $w = w_1w_2 \cdots w_m$ with $w_i \sim u_i$. As v is obtained from w by inserting a subword aa^{-1} into some w_i or appending it to the beginning or end of some w_i , v has the desired factorization. \square

Lemma 2. *Consider a fixed sequence of cancellations which reduces u to v . If two particular letters of u cancel at some point in the sequence, then either they are adjacent in u or the subword between them has been reduced to 1 by previous cancellations.*

Proof. Use induction on the length of the cancellation sequence. \square

Now we slightly generalize the definition of a pseudo-solution of equation to the following situation.

Definition 2. *A subword s of w is a pseudosolution if there is a sequence of cancellations in w which consumes all letters in s .*

We are dealing with words over Σ , not group elements. For example $s = ab^{-1}$ is a pseudosolution of $asba^{-1}a$ but not of asb . The next two

lemmas can be proved by straightforward induction on the length of an appropriate cancellation sequence.

Lemma 3. *Suppose s is a pseudosolution of $w = usv$, then $s = s_1s_2$ with s_1 a pseudosolution of us_1 and s_2 a pseudosolution of s_2v .*

Lemma 4. *Let s be a pseudosolution of w , and fix a cancellation sequence. The smallest subword of w which contains s and all letters in w canceling with letters of s is freely equal to 1.*

Lemma 5. *A subword s of w is a pseudosolution if and only if there is a word t such that s is a subword of t , t is a subword of w , and $t \sim 1$.*

Proof. If t exists, then $t \sim 1$ implies $t \rightarrow 1$ whence t and all its subwords are pseudosolutions of w . For the converse apply Lemma 4. \square

Lemma 6. *If us and sw are irreducible and if either occurrence of s is a pseudosolution of $usvsw$, then $s \sim v_3^{-1}v_1^{-1}$ for some factorization $v = v_1v_2v_3$.*

Proof. We argue by induction on n , the length of a cancellation sequence. If $n = 0$, then $s = 1$ in which case we take $u_1 = u_3 = 1$ and $u_2 = u$. If $v = 1$, then $usvsw = ussw$. As us and sw are irreducible, the only reduction possible involves cancellation at the boundary between us and sw . It follows that $ss \sim 1$, whence $s \sim 1$.

Assume $n > 0$ and $v \neq 1$. If the first reduction is within v , then $v \rightarrow p$ and by induction $s \sim p_3^{-1}p_1^{-1}$ for some factorization $p = p_1p_2p_3$. Lemma 1 implies $v = v_1v_2v_3$ with $v_i \sim p_i$ and $s \sim v_3^{-1}v_1^{-1}$.

The remaining possibilities are cancellation at the boundary between s and v or the boundary between v and s . Consider the first case; the second is similar. We have $s = ta^{-1}$ and $v = ap$ for some letter a and words t and p . The induction hypothesis applied to $utpt(a^{-1}w)$ yields $p = p_1p_2p_3$ and $t \sim p_3^{-1}p_1^{-1}$. But then $v = ap = (ap_1)p_2p_3$ and $s = ta^{-1} \sim p_3^{-1}(ap_1)^{-1}$ as desired. \square

Lemma 7. *Suppose $v \not\sim 1$. If us and $s^{-1}w$ are irreducible and s or s^{-1} is a pseudosolution of $usvs^{-1}w$, then $s \sim v''v^k$ for some integer k . (See Definition 1.) Likewise if us^{-1} and sw are irreducible and s or s^{-1} is a pseudosolution of $us^{-1}vsw$, then $s \sim v^kv'$.*

Proof. Consider the first part; as before use induction on n , the number of cancellations. If $n = 0$, then $s = 1$. Take $v_1 = v, v_2 = 1$ and $k = 0$. Otherwise the first reduction is either within v or at one end or the other of v . In the first case $v \rightarrow v'$, and the induction hypothesis applied to $usv's^{-1}w$ yields the desired result.

Suppose then that there is a reduction at the left end of v ; the other case is similar. We have $s = ta^{-1}$, $v = ap$, and application of the induction hypothesis to $ut(pa)t^{-1}w$ yields $pa = p_1p_2$ and $t \sim p_2(pa)^k$. It follows that $s \sim p_2(pa)^ka^{-1} \sim p_2a^{-1}a(pa)^ka^{-1} \sim p_2a^{-1}v^k$. If $p_2 \neq 1$, then $p_2 = v_2a$

for some suffix v_2 of v whence $s \sim v_2 v^k$. If $p_2 = 1$, then $s \sim a^{-1} v^k \sim a^{-1} v v^{k-1} \sim p v^{k-1}$. As p is a suffix of v , the first assertion holds. The second assertion follows from the first upon replacement of s by s^{-1} . \square

Lemma 8. *If s is a pseudosolution of tus , t is a pseudosolution of tvs , and st is irreducible, then $s \sim (v^{-1}u)^i(v^{-1}u)'$ and $t \sim (vu^{-1})''(vu^{-1})^j$ for some integers i, j .*

Proof. Application of Lemma 4 to tus implies either $u = u_1 u_2$ with $u_2 s \sim 1$ or $t = t_1 t_2$ with $t_2 u s \sim 1$. Consider the first case. We have $s \sim u_2^{-1} \sim (v^{-1}u)^{-1}(v^{-1}u_1)$ as required. Further $u_2^{-1} \rightarrow s$ implies that t is a pseudosolution of tvu_2^{-1} and hence of tvu^{-1} . Thus either $v = v_1 v_2$ with $t \sim v_1^{-1}$ or $u = u_3 u_4$ with $t \sim (vu_4^{-1})^{-1}$. But then $t \sim (v_2 u^{-1})(vu^{-1})^{-1}$ or $t \sim (u_3^{-1})(vu^{-1})^{-1}$, and we see that t has the right form. A similar analysis starting with starting with tvs also works.

It remains to consider the case $t = t_1 t_2$ with $t_2 u s \sim 1$ and $s = s_1 s_2$ with $tvs_1 \sim 1$. Suppose $u \sim v$. We have $t = t_1 t_2$ with $t_2 u s \sim 1$ and $s = s_1 s_2$ with $tus_1 \sim 1$. If $t_1 = 1$, then $tus \sim 1$ implies $st \sim u^{-1}$. As st is irreducible, Lemma 1 yields $s \sim (u^{-1})' = (v^{-1})'$ and $t \sim (u^{-1})''$ which is included in $i = j = 0$. If $t_1 \neq 1$, it follows from $t_2 u s_1 s_2 \sim 1 \sim t_1 t_2 u s_1$ that $t_1 \sim s_2$. As st is irreducible, t_1 and s_2 are too. Thus $t_1 = s_2 \neq 1$. Hence $s_1 s_2 t_2 = s_1 t_1 t_2$ is irreducible. But then $s_1 s_2 t_2 \sim u^{-1} \sim s_1 t_1 t_2$ implies $s \sim (v^{-1})', t \sim (u^{-1})''$ as before.

Finally suppose $t = t_1 t_2$ with $t_2 u s \sim 1$, $s = s_1 s_2$ with $tvs_1 \sim 1$, and $u \not\sim v$. From $t_2 u s \sim 1$ we deduce $u^{-1} t_2^{-1} \rightarrow s$. Hence t is a pseudosolution of $tvu^{-1} t_2^{-1}$ and all the more of $tvu^{-1} t_2^{-1} t_1^{-1} = tvu^{-1} t^{-1}$. Likewise s is a pseudosolution of $s^{-1} v^{-1} u s$. We are done by Lemma 7. \square

Lemma 9. *Let st be irreducible. If the right-hand occurrence of s is a pseudosolution in $stus$ but not in tus , then $st \sim u_3^{-1} u_1^{-1}$ for some factorization $u = u_1 u_2 u_3$. Likewise if the left-hand occurrence of t is a pseudosolution in $tust$ but not in tvs , then $st \sim v_3^{-1} v_1^{-1}$ for some factorization $v = v_1 v_2 v_3$.*

Proof. Consider the first part; the second is treated similarly. We have $s = pq$ with $q \neq 1$ and $qtus \sim 1$. Since st is irreducible, so is qt . It follows that t is a pseudosolution of tus . If t is not a pseudosolution of tu , then $s = ef$ with $tue \sim 1$. But then $qf \sim 1$ forces f to be a pseudosolution of $pqf = sf = eff$, and Lemma 6 implies $f \sim 1$. Consequently $tuef = tus \sim 1$ contrary to our hypothesis that s is not a pseudosolution of tus .

It remains to deal with the possibility that t is a pseudosolution of tu . In this case $u = u_1 u_2$ with $t \sim u_1^{-1}$. It follows that $qu_2 s \sim 1$ whence the right-hand occurrence of s is a pseudosolution in $su_2 s$. An application of Lemma 6 completes the proof. \square

Lemma 10. *Suppose $p^i u q^j \sim v$, $i, j \geq 0$, and $i|p|_c + j|q|_c \geq 2|p| + 2|q| + |u| + |v|$. (Recall Definition 1.) Further assume that p and q are not freely equal to proper powers. Under these conditions $uqu^{-1} \sim p^{-1}$.*

Proof. Assume the Lemma holds when both p and q are cyclicly reduced, and consider the case that they are not. Free reduction of p and q yields reduced words $p_1 p_2 p_1^{-1} \sim p$, $q_1 q_2 q_1^{-1} \sim q$ with p_2, q_2 cyclicly reduced. Hence $p_2^i (p_1^{-1} u q_1) q_2^j \sim p_1^{-1} v r_1$. Rewriting $p_1^{-1} u q_1$ as u_2 and $p_1^{-1} v q_1$ as v_2 we obtain $p_2^i u_2 q_2^j \sim v_2$. As $i|p_2|_c + j|q_2|_c = i|p|_c + j|q|_c \geq 2|p| + 2|q| + |u| + |v| \geq 2|p_2| + 2|u_2| + |q_2| + |v_2|$, we have $u_2 q_2 u_2^{-1} \sim p_2^{\pm 1}$. Hence $u q u^{-1} \sim (p_1 u_2 q_1^{-1})(q_1 q_2 q_1^{-1})(q_1 u_2^{-1} p_1^{-1}) \sim p_1 u_2 q_2 u_2^{-1} p_1^{-1} \sim p_1 p_2^{\pm 1} p_1^{-1} \sim p^{-1}$.

It remains to deal with the case that p and q are cyclicly reduced. Without loss of generality assume that u and v are freely reduced and $i, j \geq 0$. Thus there is a sequence of $(1/2)(|p^i u q^j| - |v|) = (1/2)(i|p| + j|q| + |u| - |v|) \geq |p| + |q| + |u|$ cancellations which reduces $p^i u q^j$ to v .

Since cancellation can occur only at either end of u , the first $|u|$ cancellations must consume u . In other words u cancels with a suffix of p^i and a prefix of q^j . For some factorizations $p = p_1 p_2$ and $q = q_1 q_2$ we have $u = (p_2 p_1^{-1})^{-1} (q_1 q_2)^{-1}$ with $i = i_1 + 1 + i_2$ and $j = j_1 + 1 + j_2$. Consequently $p^{i_1} p_1 q_2 q^{j_2}$ admits at least $|p| + |q|$ cancellations. Thus the infinite sequences $q_2 q_1 q_2 q_1 \cdots$ and $p_1^{-1} p_2^{-1} p_1^{-1} p_2^{-1} \cdots$ have the same prefix of length $|p| + |q|$. As these sequences have periods $|p|$ and $|q|$ respectively, they are identical by [8, Theorem 1]. But then the fact that $(p_1^{-1} p_2^{-1})^{|q|}$ and $(q_2 q_1)^{|p|}$ have the same length implies that they are equal. Since p and q are not proper powers, neither are $(p_1^{-1} p_2^{-1})$ and $q_2 q_1$. It follows that $p_1^{-1} p_2^{-1} = q_2 q_1$, and this equation implies in a straightforward way that $u q u^{-1} \sim p^{-1}$. \square

Lemma 11. *Suppose that q^j is a pseudosolution of $p^i u q^j v r^k$, $|j||q|_c \geq 7(|p| + |u| + |q| + |v| + |r|)$; and p, q, r are not proper powers. Then either $q \sim 1$ or $|i| \geq 1$ and $u^{-1} p u \sim q^{\pm 1}$ or $|k| \geq 1$ and $v r v^{-1} \sim q^{\pm 1}$.*

Proof. Without loss of generality assume $i, j, k \geq 0$. By Lemma 3 q factors as $q_1 q_2$ and q^j factors as $(q^{j_1} q_1)(q_2 q^{j_2})$ in such a way that $q^{j_1} q_1$ is a pseudosolution of $p^i u q^{j_1} q_1$, and $q_2 q^{j_2}$ is a pseudosolution of $q_2 q^{j_2} v r^k$. Clearly one of j_1, j_2 is no smaller than $(j - 1)/2$. Assume it is j_1 ; the argument is similar in the other case.

By Lemma 4 $q^{j_1} q_1$ extends to a suffix of $p^i u q^{j_1} q_1$ which is freely equal to 1. If that suffix is contained in $u q^{j_1} q_1$, then $u = u_1 u_2$ with $q^{j_1} \sim u_2^{-1} q_1^{-1}$. Hence q^{j_1} freely reduces to a word w with $|w| \leq |u| + |q|$. On the other hand $|w| \geq j_1 |q|_c \geq .5(j - 1)|q|_c \geq 3.5(|p| + |u| + |q|) - .5|q| \geq 3(|p| + |q| + |u|)$. But then $|p| = |q| = |u| = 0$, which implies $q \sim 1$.

It remains to consider the case that the suffix is not contained in $u q^{j_1} q_1$. In particular $i \geq 1$. For some factorization $p = p_1 p_2$ and $m \leq i$ we have $p_2 p^m u q^{j_1} q_1 \sim 1$. Thus $p^m u q^{j_1} \sim p_2^{-1} q_1^{-1}$. As above $j_1 |q|_c \geq 3(|p| + |q| + |u|) \geq 2|p| + 2|q| + |u| + |p_2^{-1} q_1|$. Lemma 10 applies and yields $u q u^{-1} \sim p^{\pm 1}$. \square

3. PARAMETRIC WORDS

In this section we show how to find a finite set of words and parametric words $uv^i w r^j$ s which together contain all solutions to Equation (1).

Let s be any freely reduced word which is a solution to Equation (1). Substitution of s for x yields a word

$$(2) \quad E(s) = u_0 s^{\varepsilon_0} \dots u_{d-1} s^{\varepsilon_{d-1}}$$

such that $E(s) \rightarrow 1$.

Fix a sequence of cancellations which reduces $E(s)$ to 1, and let s^{ε_j} be the first of the subwords $s^{\pm 1}$ to be consumed. If there is a tie, pick either subword. Observe that the letters in s^{ε_j} must cancel with nearby letters in $E(s)$. If a letter in s^{ε_j} canceled to the right of $s^{\varepsilon_{j+1}}$, then by Lemma 2 $s^{\varepsilon_{j+1}}$ would disappear before s^{ε_j} . Likewise no letter of $s^{\varepsilon_{j+1}}$ cancels to the left of $s^{\varepsilon_{j-1}}$. We have the following result.

Lemma 12. *One of the following holds.*

- (1) s^{ε_0} is a pseudosolution of $u_0 s^{\varepsilon_0} u_1 s^{\varepsilon_1}$;
- (2) For some j strictly between 0 and $d-1$, s^{ε_j} is a pseudosolution of $s^{\varepsilon_{j-1}} u_j s^{\varepsilon_j} u_{j+1} s^{\varepsilon_{j+1}}$;
- (3) $s^{\varepsilon_{d-1}}$ is a pseudosolution of $s^{\varepsilon_{d-2}} u_{d-1} s^{\varepsilon_{d-1}}$.

It is convenient to use the following immediate consequence of Lemma 12.

Lemma 13. *For some j between 0 and $d-1$, s^{ε_j} is a pseudosolution of $s^{\varepsilon_{j-1}} u_j s^{\varepsilon_j} u_{j+1} s^{\varepsilon_{j+1}}$. Here indices are read modulo d ; e.g., $u_d = u_0$.*

It follows from Lemma 13 that application of the following lemma to all successive pairs of coefficients $u = u_i$, $v = u_{i+1}$ (with indices read modulo d) yields a set of words and parametric words containing s or s^{-1} for every solution s to Equation 1.

Lemma 14. *If $\alpha, \beta = \pm 1$ and s is an irreducible pseudosolution to $s^\alpha u s v s^\beta$, then one of the following holds. (Recall Definition 1.)*

- (1) $s \sim (v^{-1}u)^i (v^{-1}u)' (vu^{-1})'' (vu^{-1})^j$;
- (2) $s \sim (u^{-1})' (u^{-1})''$ or $(v^{-1})' (v^{-1})''$;
- (3) $s \sim (u^{-1})' v^i v' v'' v^j$ or $u^i u' u'' u^j (v^{-1})''$;
- (4) $s \sim u^i u' v'' v^j$.

Proof. By Lemma 3 $s = s_1 s_2$ with s_1 a pseudosolution of $s^\alpha u s_1$ and s_2 a pseudosolution of $s_2 v s^\beta$. There are four cases. First if $\alpha = -1, \beta = -1$, Lemma 7 applied to $s_2^{-1} s_1^{-1} u s_1$ and $s_2 v s_2^{-1} s_1^{-1}$ yields (4).

If $\alpha = \beta = 1$, we have $s_1 s_2 u s_1$ and $s_1 v s_1 s_2$ where the pseudosolutions are underlined. It may happen that s_1 is pseudosolution of $s_2 u s_1$ and s_2 is a pseudosolution of $s_2 v s_1$. In this case Lemma 8 applies and (1) holds. Otherwise either s_1 is not a pseudosolution of $s_2 u s_1$ or s_2 is not a pseudosolution of $s_2 v s_1$. In both cases Lemma 9 implies (2).

Suppose $\alpha = 1, \beta = -1$. In this case $s_1 s_2 u s_1$ and $s_2 v s_2^{-1} s_1^{-1}$. By Lemma 9 either s is included in (2) or $s_2 u s_1$ whence s_1 is freely equal to the inverse of a suffix of $s_2 u$. Equivalently s_1 is freely equal to a prefix of $(s_2 u)^{-1}$. But Lemma 7 implies $s_2 \sim v_2 v^j$ for some integer j and factorization $v = v_1 v_2$.

It follows from Lemma 1 that s_1 is freely equal to a prefix of $(v_2 v^j u)^{-1}$. Consideration of the possible cases yields (3).

A similar argument works when $\alpha = -1, \beta = 1$ and shows that (2) or (3) holds. \square

4. SOLUTIONS

In order to find all solutions to Equation (1) we need to test the possibilities given by Lemma 14. It is straightforward to test the single words; the parametric words require more work. They have the form $rp^i sq^j t$. Without loss of generality we assume that p and q are not proper powers. By introducing words of the form $rp^i s$ we may assume $p \not\sim 1 \not\sim q$.

Consider $rp^i s$. Substitute rys for x in Equation 1 to obtain an equation $E'(y) = v_0 y^{\varepsilon_0} \cdots v_{d-1} y^{\varepsilon_{d-1}}$ in the indeterminate y with coefficients v_j of the form $su_j r$, $su_j s^{-1}$ etc. Note that $rp^i s$ is a solution of $E(x)$ if and only if p^i is a solution of $E'(y)$. Also the sum of the lengths of the coefficients of $E'(y)$ is $|v_0 \cdots v_{d-1}| = |u_0 \cdots u_{d-1}| + d|rs|$. Denote this number by K_1 .

If a coefficient v_j commutes with p , i.e. $v_j p \sim p v_j$, then the subword $y^{\varepsilon_{j-1}} v_j y^{\varepsilon_j}$ of $E'(y)$ may be replaced by $v_j y^{\varepsilon_{j-1} + \varepsilon_j}$ without affecting the set of i 's for which p^i is a solution. This is true even if indices are read modulo d . The coefficients in $E'(y)$ will change, but $E'(1) = v_0 \cdots v_{d-1}$ remains constant. In particular the sum of the length of the coefficients is still K_1 .

Continue replacements of this sort until reaching an equation of the form $E''(y) = w_0 y^{k_0} \cdots w_m y^{k_m}$ with m minimal. It may be that $m = 0$ and $E''(y) = w_0$. In this case p^i is a solution for all i if $w_0 = v_0 \cdots v_{d-1} \sim 1$ and for no i otherwise. Similarly if $E''(y) = w_0 y^{k_0}$, then p^i is a solution if and only if $w_0 \sim p^{-ik_0}$. In this case the free reduction of p^{ik_0} is a word of length at least $|ik_0||p|_c$ and at most $|w_0| = K_1$. Consequently $|i||p|_c \leq |ik_0||p|_c \leq K_1$.

The remaining possibility is that $E''(y) = w_0 y^{k_0} \cdots w_m y^{k_m}$ with $m \geq 2$, all $k_j \neq 0$ and no w_j commuting with p . No w_j conjugates p to p^{-1} either, as p and p^{-1} are not conjugate in the free group F . If p^i is a solution, then by Lemma 13 (with $E''(y)$ in place of $E(x)$) some p^{ik_j} must be a pseudosolution of $p^{ik_{j-1}} w_j p^{ik_j} w_{j+1} p^{ik_{j+1}}$. Lemma 11 now implies that $|i||p|_c < 7(|p| + |w_j| + |p| + |w_{j+1}| + |p|) \leq 21|p| + 7K_1$. We have proved the following lemma.

Lemma 15. *If $rp^i s$ is a solution to $E(x) = 1$ for some i with $|i||p|_c > 21|p| + 7(|u_0 \cdots u_{d-1}| + d|rs|)$, then $rp^i s$ is a solution for all i .*

Consider a solution $rp^i sq^j t$ to $E(x) = 1$. Define $K_2 = 21 \max\{|p|, |q|\} + 7(|u_0 \cdots u_{d-1}| + d|rst|)$. We will show that either $|i||p|_c$ or $|j||q|_c$ is no larger than $K_2 d$. Thus each parametric word $rp^i sq^j t$ from Lemma 12 with two parameters may be replaced by a collection of parametric words with just one parameter, namely $rp^{i_0} sq^j t$, $rp^i sq^{j_0} t$ with $|i_0||p|_c \leq K_2 d$ and $|j_0||q|_c \leq C_2 d$.

Without loss of generality suppose that $i, j \geq 0$, and p and q are not proper powers. In particular the centralizers in the free group F of p and q are the cyclic subgroups generated by p and q respectively.

Lemma 16. *Suppose p is not conjugate to q or q^{-1} and $rp^i sq^j t$ is a solution to $E(x) = 1$. Then either $|i||p|_c$ or $|j||q|_c$ is no larger than $K_2 = 21 \max\{|p|, |q|\} + 7(|u_0 \cdots u_{d-1}| + d|rst|)$.*

Proof. First suppose that $s \sim 1$ and take the solution to be $rp^i q^j t$. Write $E(rp^i q^j s) = v_0(p^i q^j)^{\varepsilon_1} v_1 \cdots v_{d-1}(p^i q^j)^{\varepsilon_d} v_d$. The v_k 's are coefficients; call the $(p^i)^{\varepsilon_k}$'s and $(q^j)^{\varepsilon_k}$'s powers. Consider how a coefficient v_k might conjugate the power on one side of itself to the power on the other side. As p is not conjugate to q or q^{-1} , v_k would either lie in a subword $v_{k-1}p^i q^j v_k q^{-j} p^{-i} v_{k+1}$ and centralize q or in a subword $v_{k-1}q^{-j} p^{-i} v_k p^i q^j v_{k+1}$ and centralize p . Consequently v_k is freely equal to a nontrivial power (because $E(x)$ is freely reduced) of q in the first case and a nontrivial power of p in the second. W is freely equal to the word obtained by deleting the powers on either side of v_k .

Let W' be the word obtained from W by performing all the deletions discussed in the previous paragraph. Notice that the first and last powers of W survive and that the new coefficients are either old coefficients which do not conjugate their adjacent powers into each other or products $v_k v_{k+1} \cdots v_{k+m}$ of successive coefficients whose adjacent powers in W have been deleted. In the latter case the coefficient is an alternating product of nontrivial powers of p and q .

Since $W' \sim 1$, some power is a pseudosolution in a subword of W' consisting to up to three powers and the coefficients between them. The sum of the length of the coefficients of W' is the same as that of W , namely $\sum |u_k| + d|r| + d|s|$. If $|i||p|_c$ and $|j||q|_c$ exceed the bound given above, then Lemma 11 applies and (as p is not conjugate to q or q^{-1}) implies that some coefficient conjugates one adjacent power to the other. But this is impossible either because the coefficient is inherited from W or because the coefficient is an alternating product of nontrivial powers of p and q , and the conjugation would be a nontrivial relation satisfied by p and q , which generate a free group of rank two.

It remains to reduce to the case $s \sim 1$. Assume $s \not\sim 1$, and rewrite the solution as $rp^i (sq s^{-1})^j (st)$. One of $|i||p|_c$ or $|j||sq s^{-1}|_c = |j||q|_c$ is at most $21 \max\{|p|, |sq s^{-1}|\} + 7(|u_0 \cdots u_{d-1}| + d|rst|)$. \square

Finally, consider a solution $rp^i sq^j t$ to $E(x) = 1$ with p conjugate to q or q^{-1} . With appropriate changes to r, s, t and j , $rp^i sq^j t$ may be rewritten as $rp^i sp^j t$ where p is cyclicly reduced and s does not commute with p .

Define $W = E(rp^i sp^j t) = v_0(p^i sp^j)^{\varepsilon_1} v_1 \cdots v_{d-1}(p^i sp^j)^{\varepsilon_d} v_d$, and argue as before. The coefficients now include the subwords $s^{\pm 1}$ as well as the v_k 's. Consider how a v_k might conjugate the power on one side of itself to the power on the other side. Since all powers are powers of p , v_k would commute with p and hence would itself be freely equal to a power of p . If $\varepsilon_k \varepsilon_{k+1} = -1$, then $v_k \not\sim 1$ and the powers on either side cancel. However, if $\varepsilon_k \varepsilon_{k+1} = 1$, then the powers do not necessarily cancel but combine to form a power $p^{\pm(i+j)}$.

Let W' be the word obtained from W by performing all the deletions and combinations of powers discussed in the previous paragraph. Notice that the first and last powers of W survive and that the new subwords between powers surviving from W are either coefficients from W which do not conjugate their adjacent powers into each other or alternating products $s^{\pm 1}p^{k_m}v_ms^{\pm 1}p^{k_{m+1}}v_{m+1}\dots s^{\pm 1}p^{k_n}v_ns^{\pm 1}s$ where the v_j 's which occur are freely equal to powers of p . Further if p^{k_j} occurs between s and s^{-1} , then $k_j = 0$ and v_j is freely equal to a nontrivial power of p , while if p^{k_j} occurs between two s 's or two s^{-1} 's, then $k_j = \pm(i+j)$.

There are two possibilities. First if $|(i+j)||p|_c \geq K_2$, then we may consider the subwords $p^{\pm(i+j)}$ to be powers like the $p^{\pm i}$'s and $p^{\pm j}$'s surviving from W and the subwords between the powers as coefficients. Lemma 11 applies and implies that some coefficient conjugates one adjacent power to the other and hence is a power of p . But this is impossible either because the coefficient is inherited from W or because the coefficient is an alternating product of nontrivial powers of p and s , and the conjugation would be a nontrivial relation satisfied by the subgroup generated by p and s , which is free of rank two.

Second if $|(i+j)||p|_c < K_2$, we take just the $p^{\pm i}$'s and $p^{\pm j}$'s from W to be powers. The coefficients are either inherited from W or alternating products $s^{\pm 1}p^{k_m}v_ms^{\pm 1}p^{k_{m+1}}v_{m+1}\dots s^{\pm 1}p^{k_n}v_ns^{\pm 1}s$ as above. In this case $|(i+j)||p| = |(i+j)||p|_c \leq K_2$, and the total length of the coefficients increases to at most $K_2 + (d-1)|(i+j)||p| \leq dK_2$. Lemma 11 applies and yields the following lemma.

Lemma 17. *Suppose p is conjugate to q or q^{-1} and rp^isq^jt is a solution to $E(x) = 1$. Then $|i||p|_c$ or $|j||q|_c$ is no larger than dK_2 .*

5. THE ALGORITHM

The algorithm implicit in the preceding analysis may be described as follows.

- (1) The input is an equation $u_0x^{\varepsilon_0}u_1x^{\varepsilon_1}\dots u_{d-1}x^{\varepsilon_{d-1}} = 1$ of degree $d \geq 2$ and with freely reduced coefficients from a free monoid Σ^* over a set Σ of generators and their inverses for a free group F .
- (2) Let L be the list of words and parametric words and their inverses from Lemma 14. Rewrite the parametric words so that they are either ordinary words or have the one of the forms rp^is or rp^isq^jt with $p \not\sim 1 \not\sim q$, p, q not proper powers, and in the latter case $sq s^{-1} \not\sim p^{\pm 1}$.
- (3) For each ordinary word $w \in L$ test $E(w) \sim 1$ and $E(w^{-1}) \sim 1$. Remove w from L .
- (4) Replace each parametric word rp^isq^jt with words $rp^isq^{j_0}t$ and $rp^{i_0}sq^jt$ for all i_0, j_0 with $|i_0||p|_c \leq dK_2$ and $|j_0||q|_c \leq dK_2$ where K_2 is as in Lemma 16.

- (5) For each word of the form $w = rp^i q$ in L , if $E(rp^{i_0} s) \sim 1$ where i_0 is the least integer greater than $(1/|p|_c)(21|p| + 7(|u_0 \cdots u_{d-1}| + d|rs|))$, then $x = rp^i s$ is a solution for all i , otherwise test $E(rp^{i_1} s) \sim 1$ for all $|i_1| < i_0$.

We leave it to the reader to check that our preceding analysis implies the correctness of the above algorithm. To bound the time complexity let $|L|$ be the length the list from Step 2 and M the maximum of $|rpsqt|$ for each entry $rp^i sq^j t$. Note that M is also an upper bound for the length of the coefficients of $E(x)$ and that the constant K_2 from Lemma 16 is $O(dM)$.

Steps 2 and 3 are accomplished in time $O(M|L|)$, and Step 4 in time $O(dK_2 M|L|)$. Let L' be the augmented list from Step 4. $|L'| = O(dK_2|L|) = O(d^2 M|L|)$, and each entry in L' has the form $rp^i s$ with $|rps| = O(dMK_2) = O(d^2 M^2)$.

For each entry there are $O(M + dM + d^2 MK_2) = O(d^3 M^2)$ tests performed in Step 5. The time to test the entry $rp^i s$ is linear in the length of $E(rp^i s)$, which is $O(d|rp^i s| + |u_1 \cdots u_{d-1}|) = O(id^3 M^2) = O(dK_2 d^3 M^2) = O(d^5 M^3)$. Thus the total time for Step 5 is $O(|L'| \cdot (d^3 M^2) \cdot (d^5 M^3)) = O((d^2 M|L|) \cdot (d^3 M^2) \cdot (d^5 M^3)) = O(d^{10} M^6 |L|)$. Clearly this estimate bounds the time of the complete algorithm.

Finally let m be the maximum size of a coefficient in $E(x)$. It follows from Lemma 14 that $M = O(m)$ and that $|L| = O(dm^3)$. Thus the time complexity of our algorithm is $O(d^{11} m^9)$.

REFERENCES

- [1] K. Appel, *One-variable equations in free groups*, Proc. Amer. Math. Soc., **19** (1968) 912–918.
- [2] G. Baumslag, A. Myasnikov, V. Remeslennikov. *Algebraic geometry over groups I. Algebraic sets and ideal theory*. Journal of Algebra, 1999, v.219, pp.16–79.
- [3] G. Baumslag, A. Myasnikov, V. Remeslennikov. *Discriminating completions of hyperbolic groups*, Geometriae Dedicata (2003), Vol 92, pp.115–143.
- [4] D. Bormotov. *A package of algorithms for solving hard problems in group theory*, Dissertation Thesis, The Graduate School of CUNY, New York, 2005.
- [5] I. Bumagin, O. Kharlampovich, A. Myasnikov *Isomorphism problem for finitely generated fully residually free groups*. J. of Pure and Applied Algebra, 2006.
- [6] I. Chiswell and V. N. Remeslennikov, *Equations in free groups with one variable I*, J. Group Theory, **3** (2000) 445–466.
- [7] V. Diekert, C. Gutierrez, C. Hagenah *The existential theory of equations with rational constraints in free groups is PSPACE-complete*,
- [8] N. J. Fine and H. S. Wilf, *Uniqueness theorems for periodic functions*, Proc. Amer. Math. Soc. **16** 1965 109–114.
- [9] R. Gilman and A. G. Myasnikov, *One variable equations in free groups via context free languages*, in Contemporary Mathematics, **349** (2004) 83–88.
- [10] C. Gutierrez. *Satisfiability of equations in free groups is in PSPACE*. In 32nd Ann. ACM Symp. on Theory of Computing (STOC'2000), pages 21–27. ACM Press, 2000.
- [11] O. Kharlampovich, A. Myasnikov. *Irreducible affine varieties over a free group. I: Irreducibility of quadratic equations and Nullstellensatz*. J. of Algebra, 1998, v. 200, n. 2, p.472–516.

- [12] O.Kharlampovich and A.Myasnikov *Effective JSJ decompositions*, Contemp.Math. AMS, Algorithms,Languages, Logic (Borovik, ed.), CONM/ 378, 2005, p.87-212.
- [13] O.Kharlampovich, A. Myasnikov. *Implicit function theorems over free groups*. Journal of Algebra, 290/1 (2005), p.1-203.
- [14] O.Kharlampovich, A. Myasnikov. *Elementary theory of free nonabelian groups*. Journal of Algebra, 2006.
- [15] A. Kościelski, L. Pacholski. *Makanin's algorithm is not primitive recursive*, Theoretical Computer Science 191, pp. 145-156, 1998.
- [16] A. Kvaschuk, A. Myasnikov, *Big powers and free constructions*, to appear in International Journal of Algebra and Computation.
- [17] R. C. Lyndon, *Equations in free groups*, Trans. Amer. Math. Soc., **96** (1960) 445–457.
- [18] ———, *Groups with parametric exponents*, Trans. American Math. Soc., **96** (1960) 518-533.
- [19] A.A. Lorents, *The solution of systems of equations in one unknown in free groups*, Dokl. Akad. Nauk **148** (1963) 1253-1256.
- [20] ———, *Representations of sets of solutions of systems of equations with one unknown in a free group*, Dokl. Akad. Nauk **178** (1968) 290-292.
- [21] G.S. Makanin. *Equations in a free group*. Izvestiya NA SSSR 46, pp. 1199-1273, 1982 (in Russian). English translation in Math USSR Izvestiya, Vol.21, No.3, 1983.
- [22] A.Myasnikov, V.Remeslennikov. *Algebraic geometry 2: logical foundations*. Journal of Algebra 234 (2000), p. 225-276.
- [23] A. Myasnikov and V. Remeslennikov. *Exponential groups 2: extension of centralizers and tensor completion of CSA-groups*. Intern. Journal of Algebra and Computation, Vol.6, No.6 (1996), p.687-711.
- [24] W. Plandowski. *Satisfiability of Word Equations with Constants is in PSPACE*, in. Proc. FOCS'99.
- [25] A.A. Razborov, *On systems of equations in a free group*, Izvestiya AN SSSR 48, pp. 779-832, 1984 (in Russian). English translation in Math. USSR Izvestiya 25, pp. 115-162, 1985.
- [26] A.A. Razborov, *An equation in a free group whose set of solutions does not allow a representation as a superposition of a finite number of parametric functions*. In Proceedings of the 9th All-Union Symposium on the Group Theory, Moscow, 1984, p.54.